Some intro on thermal operations

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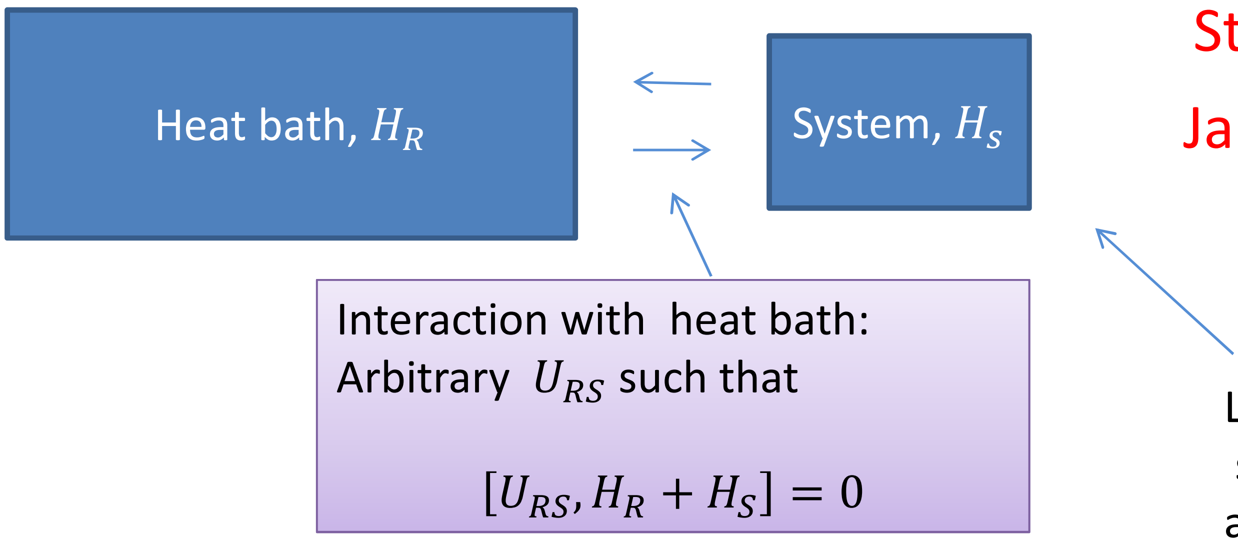
Thermal Operations

Streater, book 1995

Janzing et al. 2000



Later we will
split it into system S
and work system W

First Law: ensured by $[U_{RS}, H_R + H_S] = 0$

Second Law: ensured by unitarity of U_{RS}.

Remark: In our approach, drawing work will be defined by state change!

Questions:

- Find condition for general transition $(\rho_S, H_S) \rightarrow (\sigma_S, H_S)$
- Define work, derive optimal work extraction

Theorem[HO2011]: For diagonal states,

state (ρ, H) can be transformed into (σ, H) by thermal operations

if and only if

$f_\rho(x) \geq f_\sigma(x)$ for all $x \in [0, Z]$.

Actually: immediate conclusion from Ruch& Mead 1976 + Janzing et al. 2000]

Thermo-diagrams (Lorentz curves)

state (ρ, H) ⟷ function $f_\rho : [0, Z] \to [0,1]$

energy ↘ $p(E)$ – eigenvalues of ρ

here Z is partition function of H

1) Rescale: $p'(E) = p(E)/e^{-\beta E}$

2) Order (decreasingly): $\frac{p(E_1)}{e^{-\beta E_1}} \geq \frac{p(E_2)}{e^{-\beta E_2}} \geq \cdots \geq \frac{p(E_n)}{e^{-\beta E_n}}$

3) f is linear interpolation of points

$(0,0)$, $(e^{-\beta \mathrm{E}_1}, p_1)$, $(e^{-\beta \mathrm{E}_1} + e^{-\beta \mathrm{E}_2}, p_1 + p_2)$, … , $(Z, 1)$

Thermo-diagrams (Lorentz curves)

Order (decreasingly): $\frac{p(E_1)}{e^{-\beta E_1}} \geq \frac{p(E_2)}{e^{-\beta E_2}} \geq \cdots \geq \frac{p(E_n)}{e^{-\beta E_n}}$

Recall:
p_i's are not ordered!
they are β - ordered

Denote: $p_i = p(E_i)$

3) f is linear interpolation of points:

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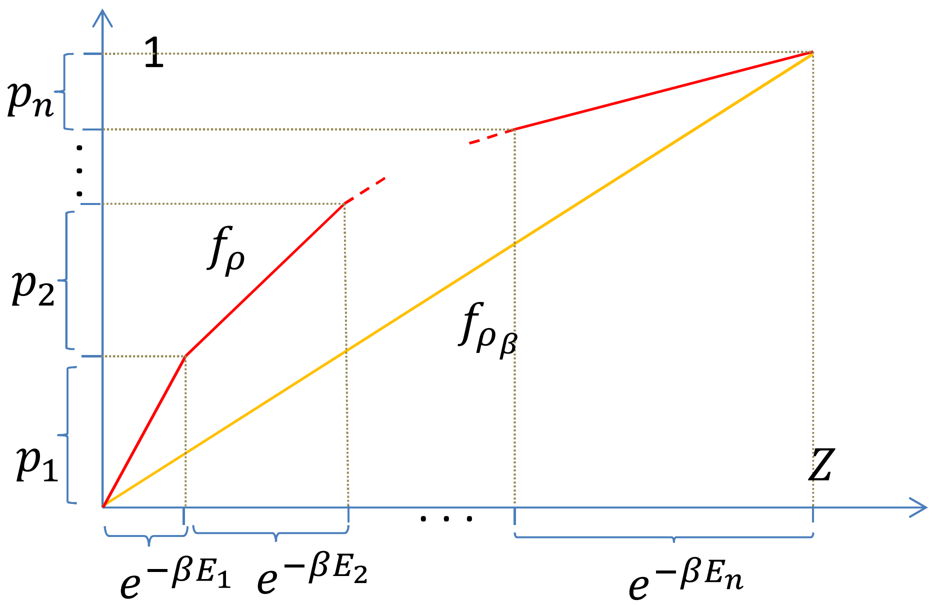
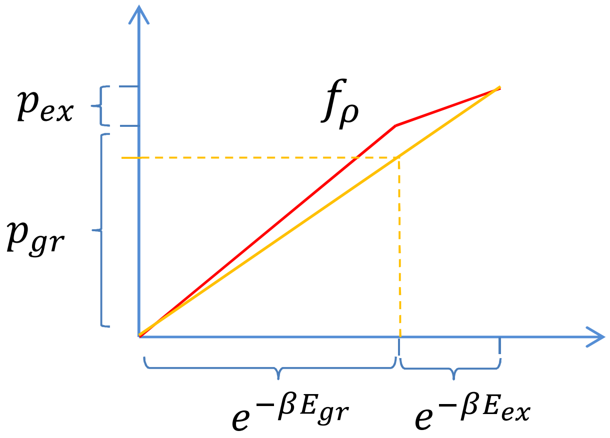
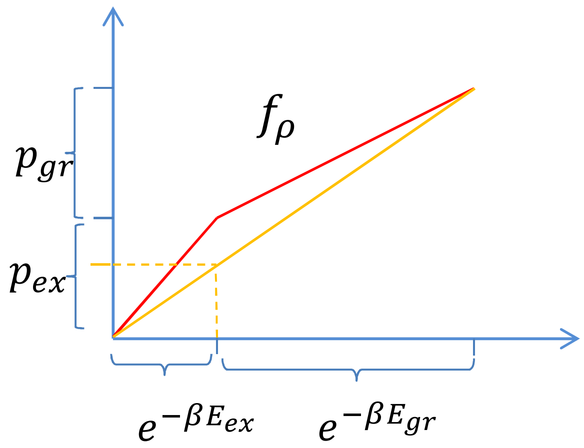


Thermo-diagrams for two level systems

Two β-orders possible:

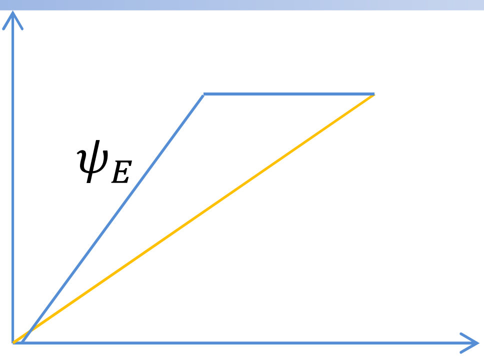


ρ is less excited
than Gibbs state



ρ is more excited
than Gibbs state

The form of diagram of eigenstate of energy:



Transition Law for diagonal states: Thermomajoriation

Theorem[H&O Nat Com]:
State (ρ, H) can be transformed into (σ, H) by thermal operations

$$(\rho, H) \xrightarrow{TO} (\sigma, H)$$

if and only if

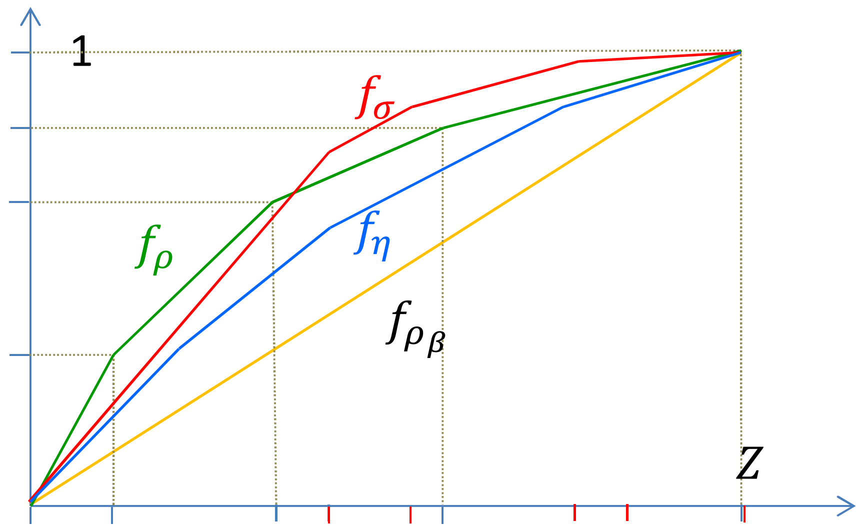
$$f_\rho(x) \geq f_\sigma(x)$$

for all $x \in [0, Z]$.

Actually: immediate conclusion from Ruch& Mead 1976 + Janzing et al. 2000]

Thermomajoriation criterion

Theorem[H&O Nat Com]: state (ρ, H) can be transformed into (σ, H) by thermal operations if and only if $f_\rho(x) \geq f_\sigma(x)$ for all $x \in [0, Z]$.



$\rho \nleftrightarrow \sigma$

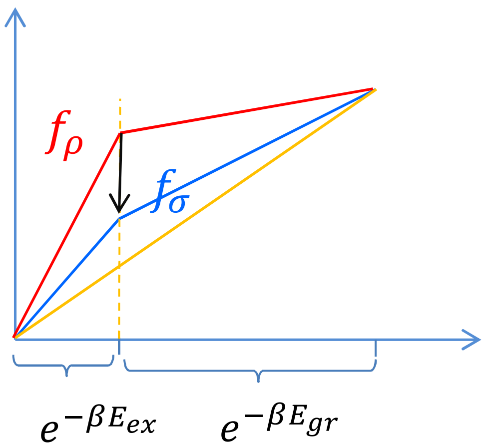
$\rho \rightarrow \eta$

$\sigma \rightarrow \eta$

$\eta \nrightarrow \sigma$

$\eta \nrightarrow \rho$

Two level system: the same β-order



Example:
For ρ and σ more excited than Gibbs state:

$$\rho \rightarrow \sigma \quad \Leftrightarrow p_{ex}(\rho) \geq p_{ex}(\sigma)$$

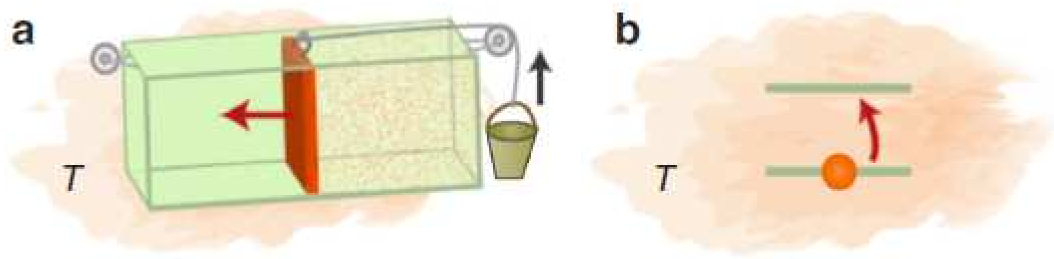
Remark:
This transition can be done by just **mixing** with Gibbs state

Questions:

- Find condition for general transition $(\rho_S, H_S) \rightarrow (\sigma_S, H_S)$
- Define work, derive optimal work extraction

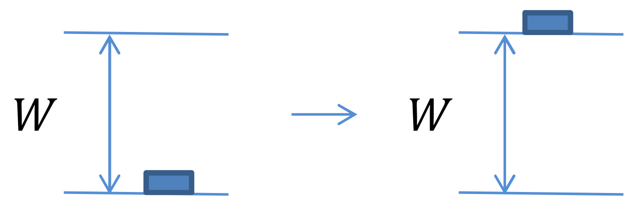
Definition of deterministic work

[H&O Nat Com]



Macroscopic work Microscopic work

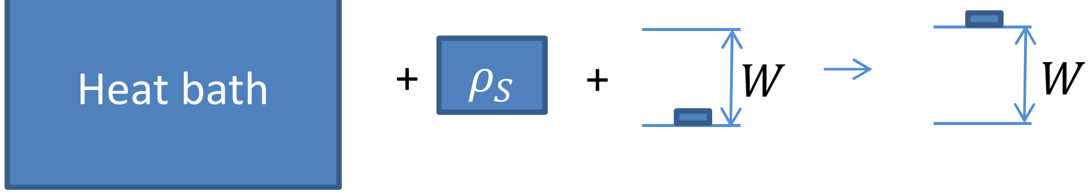
We perform work W if we transform eigenstate of energy E to eigenstate of energy E', with $E' - E = W$



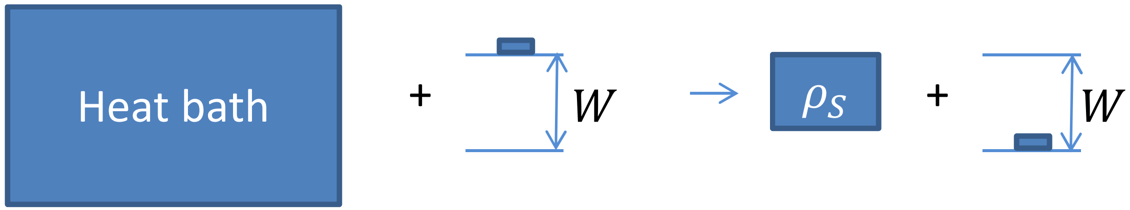
Fact: Without loss of generality, one can restrict to two level work system, which we call **Wit (Work-bit)**

Basic questions related to work

- $\rho \rightarrow W$ (maximal **extractable** work)

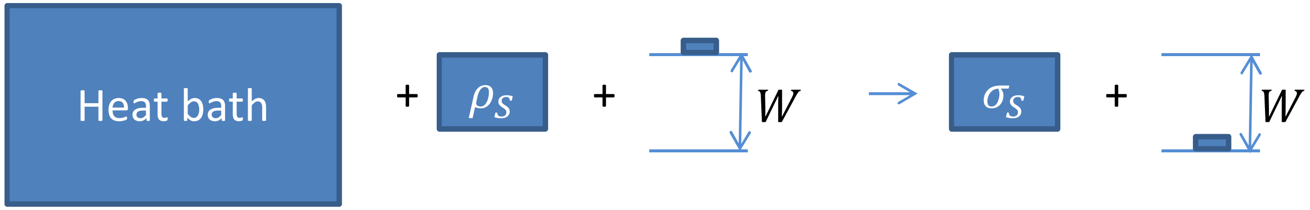


- $W \rightarrow \rho$ (minimal work needed to **form** state)



Most general question:

- $\rho + W \rightarrow \sigma$



A primitive: work rescaling

How are related diagrams of
- $\rho_S \otimes |W\rangle\langle W|$
- $\rho_S \otimes |0\rangle\langle 0|$
- ρ_s itself

Domains:
- $f_{\rho\otimes|W\rangle\langle W|}$: $[0, Z_S Z_W] \to [0,1]$
- $f_{\rho\otimes|0\rangle\langle 0|}$: $[0, Z_S Z_W] \to [0,1]$
- f_ρ: $[0, Z_S] \to [0,1]$

Let E_1, E_2, E_3 be β-ordered energies with respect to ρ

$$\underbrace{E_1, E_2, E_3}_{\text{occupied by } \rho_S \otimes |0\rangle\langle 0|}, \underbrace{E_1 + W, E_2 + W, E_3 + W}_{\text{occupied by } \rho_S \otimes |W\rangle\langle W|}$$

⇒

$$f_{\rho\otimes|0\rangle\langle 0|}(x) = \begin{cases} f_\rho(x) & x \in [0, Z_S] \\ 1 & else \end{cases}$$

$$f_{\rho\otimes|W\rangle\langle W|}(x) = \begin{cases} f_\rho\left(xe^{-\beta W}\right) & x \in [0, Z_S e^{-\beta W}] \\ 1 & else \end{cases}$$

Conclusion:
- All diagrams nontrivial only for $x \in [0, Z_S]$
- The diagram of $\rho_S \otimes |W\rangle\langle W|$ is obtained from that of ρ_S by **rescaling** x-axis .
- The diagram of $\rho_S \otimes |0\rangle\langle 0|$ is **the same** as that of ρ_S

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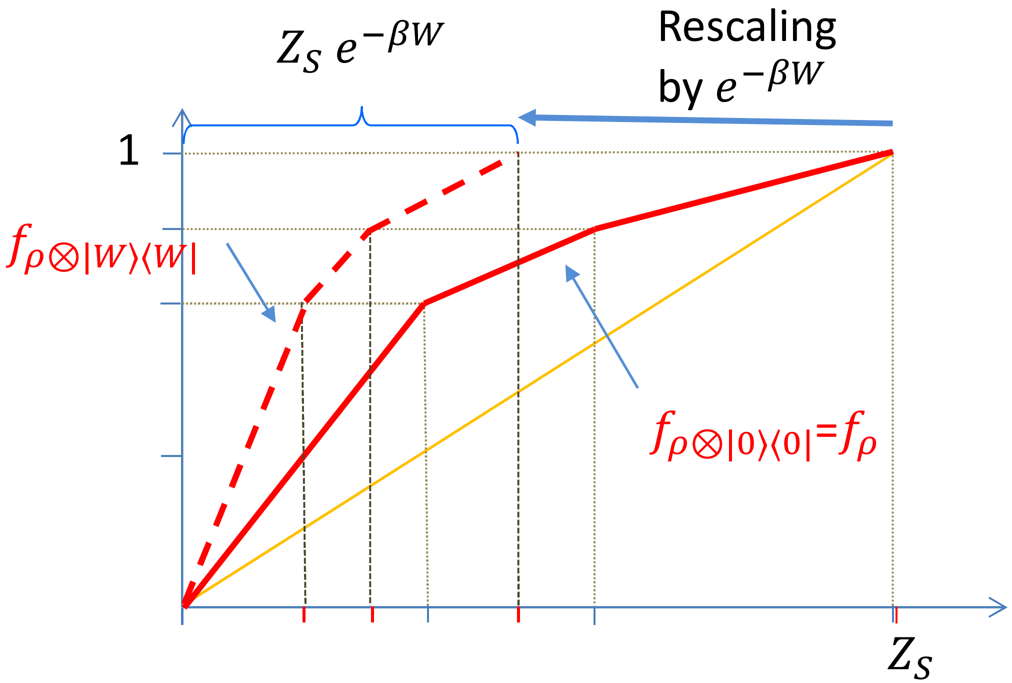
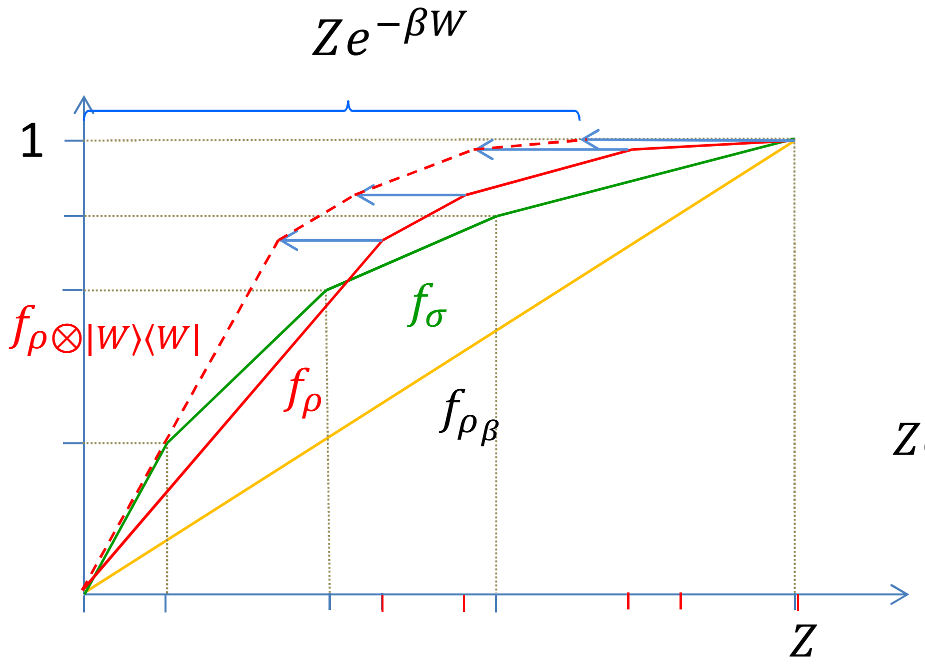


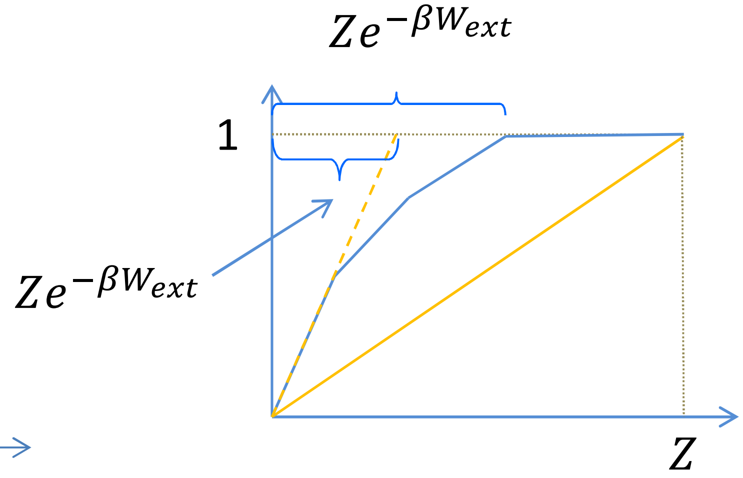
Law of transitions including work

General case: $\rho + W \rightarrow \sigma$



Special case:

- work extraction $\rho \rightarrow W_{\text{ext}}$
- formation $\text{W}_{\text{form}} \rightarrow \sigma$



Theorem [HO]: minimal work needed to transform ρ into σ is given by:

$$W(\rho \rightarrow \sigma) = \min\{W: f_\rho\left(xe^{-\beta W}\right) \geq f_\sigma(x)\}$$

In particular:

$$W_{ext} = \Delta F_{min}$$
$$W_{form} = \Delta F_{max}$$

PUZZLE

Work is defined in terms of state change. But everything we can do to the state is determined by thermomajorization.

Therefore one must be able to derive fluctuation relations from thermal operations.

But: so far we considered only deterministic work snd only two possible values of work

Let us be more general: apply paradigm of thermal operations to
a) consider non-deterministic work -> distribution of work
b) consider other batteries than wit

Puzzle: if we thermalize wit, we obtain positive work gain seems to contradict to fluctuation theorem.

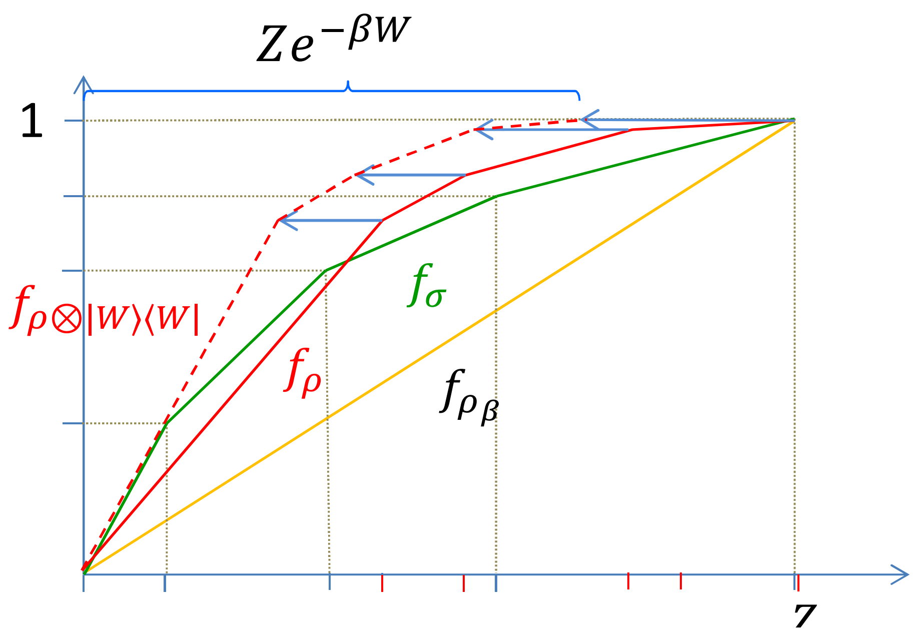
Solution: Alhambra, Masanes, Oppenheim and Perry. -> Translational invariant battery – weight.

But, then Wit in trouble ☹

Using wit we get things not allowed by fluctuation relations.

Can we now treat seriously the formula below????

$$W(\rho \to \sigma) = \min\{W : f_\rho\left(xe^{-\beta W}\right) \geq f_\sigma(x)\}$$



Looks like our results were ruined in front of our eyes...

We sent Patryk Lipka-Bartosik for the expedition to


Save the Wit!
30/10/2013

Thermo-majoriation for two-level systems

Two level system: the same β-order

Case (1):
For ρ and σ more excited than Gibbs state:

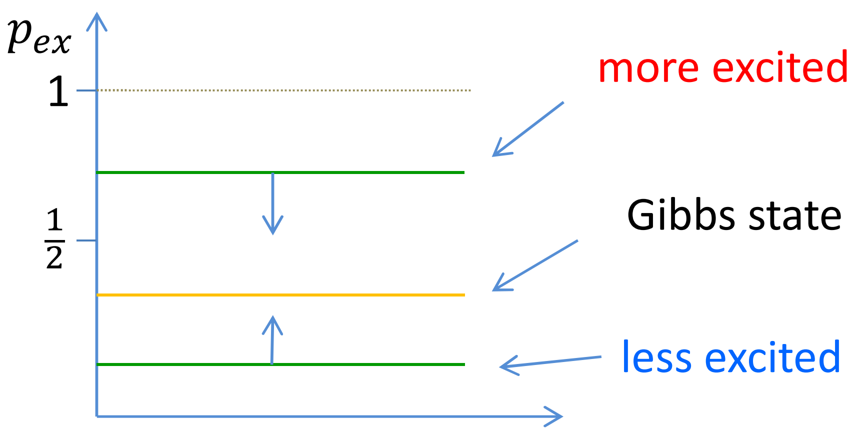
$$\rho \to \sigma \quad \Leftrightarrow p_{ex}(\rho) \geq p_{ex}(\sigma)$$

i.e. we can **dexcite** towards Gibbs state

Case (2):
For ρ and σ less excited than Gibbs state:

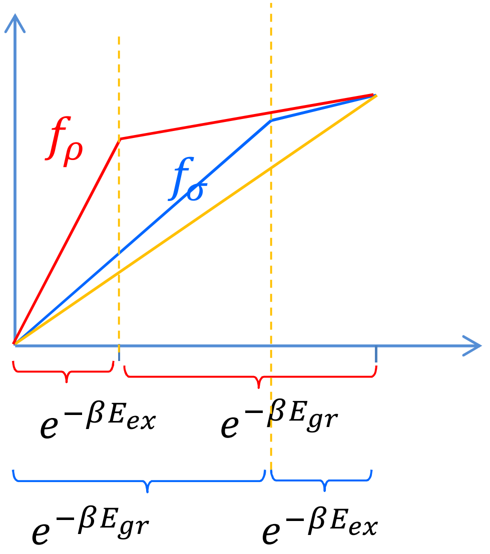
$$\rho \to \sigma \quad \Leftrightarrow p_{ex}(\rho) \leq p_{ex}(\sigma)$$

i.e. we can **excite** towards Gibbs state

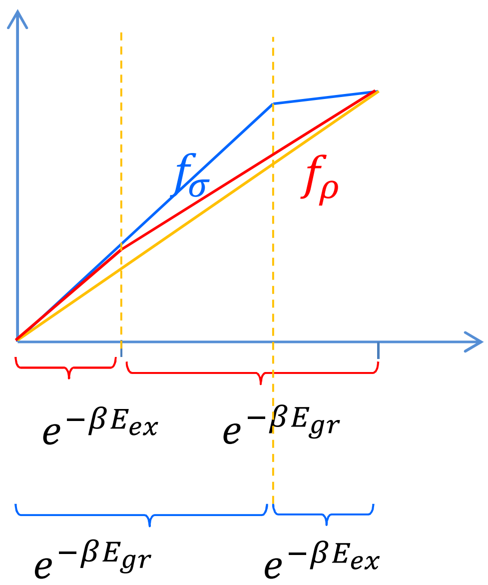


Thermo-majoriation for two-level systems

Two level system: different β-order



more excited than Gibbs → less excited than Gibbs



less excited than Gibbs → more excited than Gibbs