# Extremal points of thermal processes

P. Mazurek, M. Horodecki
M. Lostaglio, A. Alhambra, C. Perry

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# Inspiration – Elementary Thermal Operations [1]

Thermal Operations

$T(\rho) = Tr_B[U\left(\rho \otimes \frac{e^{-\beta H_B}}{Tr[e^{-\beta H_B}]}\right)U^\dagger]$

$[U, H_S + H_B] = 0$, $H_B = \sum_i E_i |i\rangle\langle i|$.

$\beta = \frac{1}{kT_B}$.

Bringing arbitrary systems in $\beta$
Discarding subsystems

Thermal Processes

$\mathbf{T}_{i,j} = \langle j|T(|i\rangle\langle i|)|j\rangle$:

$\mathbf{T}g = g$, $\bigwedge_j \sum_i \mathbf{T}_{i,j} = 1$

$g = (1, q_{10}, \dots, q_{d-1,0})\frac{1}{Z}, Z = 1 + q_{10} \dots + q_{d-1,0}$

$q_{nm} = e^{-\beta(E_m - E_n)}$.

Elementary Thermal Processes (ETP)

1. $\mathbf{E}$ acts non-trivially only on two levels i,j.
2. $\mathbf{E}_{i,j} = \mathbf{E}_{j,i} q_{ij}$.

Are there thermal operations than cannot be performed by Elementary Thermal Operations (ETO)?

-> Yes. For a d-dimensional system, one can find a pair of states $p = (p_0, \dots, p_{d-1})$ and $r = (r_0, \dots, r_{d-1})$ and unique $\mathbf{T}$ such that $\mathbf{T}p = r$, but $\mathbf{T}$ cannot be decomposed into a product of thermal operations, each acting on at most $d-1$ dimentional subspaces.

[1] M. Lostaglio, arXiv:1607.00394

# Thermal operation undecomposible into ETO

$$T(p) = r\ , p = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, r = \begin{pmatrix} 1 - \sum_{i>0} q_{i0} \\ q_{10} \\ \vdots \\ q_{d-1,0} \end{pmatrix}.$$

We have $T_{0,0} \ = 1 - \sum_{i>0} q_{i0},$ and Gibbs preserving condition: $\sum_j T_{i,j}\, q_{j0} \ = q_{i0}$ implies (for $i = 0$) $\Lambda_{j>0} T_{0,j} \ = 1$ . Then:

$$T = \begin{pmatrix} 1 - \sum_{i>0} q_{i0} & 1 & \cdots & 1 \\ q_{10} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ q_{d-1,0} & 0 & \cdots & 0 \end{pmatrix}.$$

We will show that for $T = AB$, one of $\{A, B\}$ is equal to $T$, while another acts trivially on the ground level. This implies that every decomposition into $k$ processes has the form $T = X_1 \dots X_n T Y_1 \dots Y_{k-n-1}$, with $n \in (0, \dots k - 1)$ and $X_i$, $Y_i$ acting trivially on the ground level.

$q_{nm} = e^{-\beta(E_m - E_n)}$.

# Thermal operation undecomposible into ETO

$$T = \begin{pmatrix} 1 - \sum_{i>0} q_{i0} & 1 & \cdots & 1 \\ q_{10} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ q_{d-1,0} & 0 & \cdots & 0 \end{pmatrix}$$

$$T = AB \;\Rightarrow\; \forall_{i>0,j>0}\, T_{i,j} = 0 \;\Rightarrow \forall_{i>0,j>0}\, A_{i,0} B_{0,j} = 0$$

Assume $\exists_{k>0} B_{0,k} \neq 0$.
Then $\forall_{i>0} A_{i,0} = 0 \Rightarrow A_{0,0} = 1 \wedge \forall_{i>0} A_{0,i} = 0 \Rightarrow B = T$.

Assume $\nexists_{k>0} B_{0,k} \neq 0$.
Then $B_{0,0} = 1 \wedge \forall_{i>0} B_{i,0} = 0 \Rightarrow A = T$.

$q_{nm} = e^{-\beta(E_m - E_n)}$.

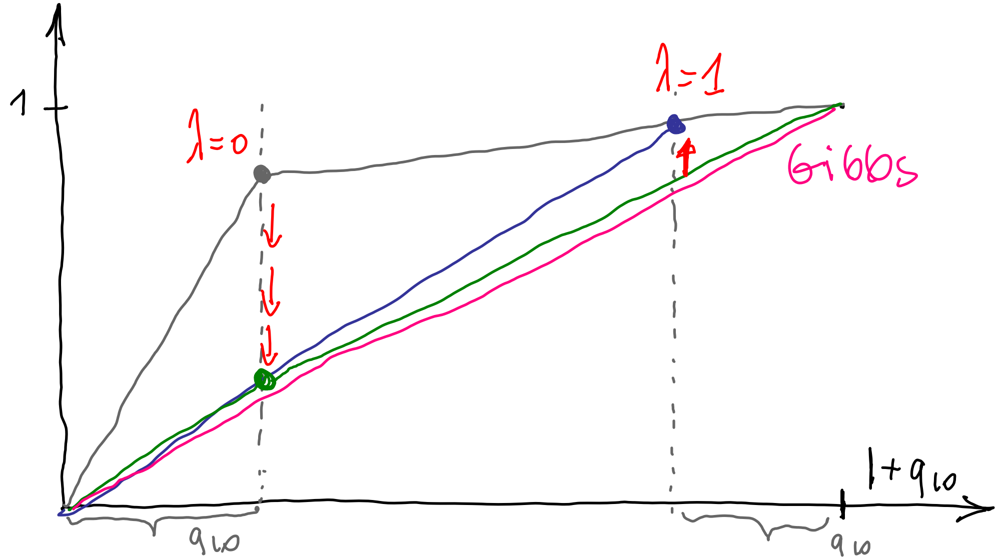
# Extremal points of thermal processes

$$T = \begin{pmatrix} 1 - q_{10} - q_{20} & 1 & 1 \\ q_{10} & 0 & 0 \\ q_{20} & 0 & 0 \end{pmatrix}$$

$d^2$elements, $2d-1$ independent constraints ⇒ at least $(d-1)^2$ zero elements

Always 1 column with identity and one row with a single non-zero element ($q_{mn}$). Therefore, there will be at least one slope at the majorization diagram, that will be conserved during the action of this extremal point:

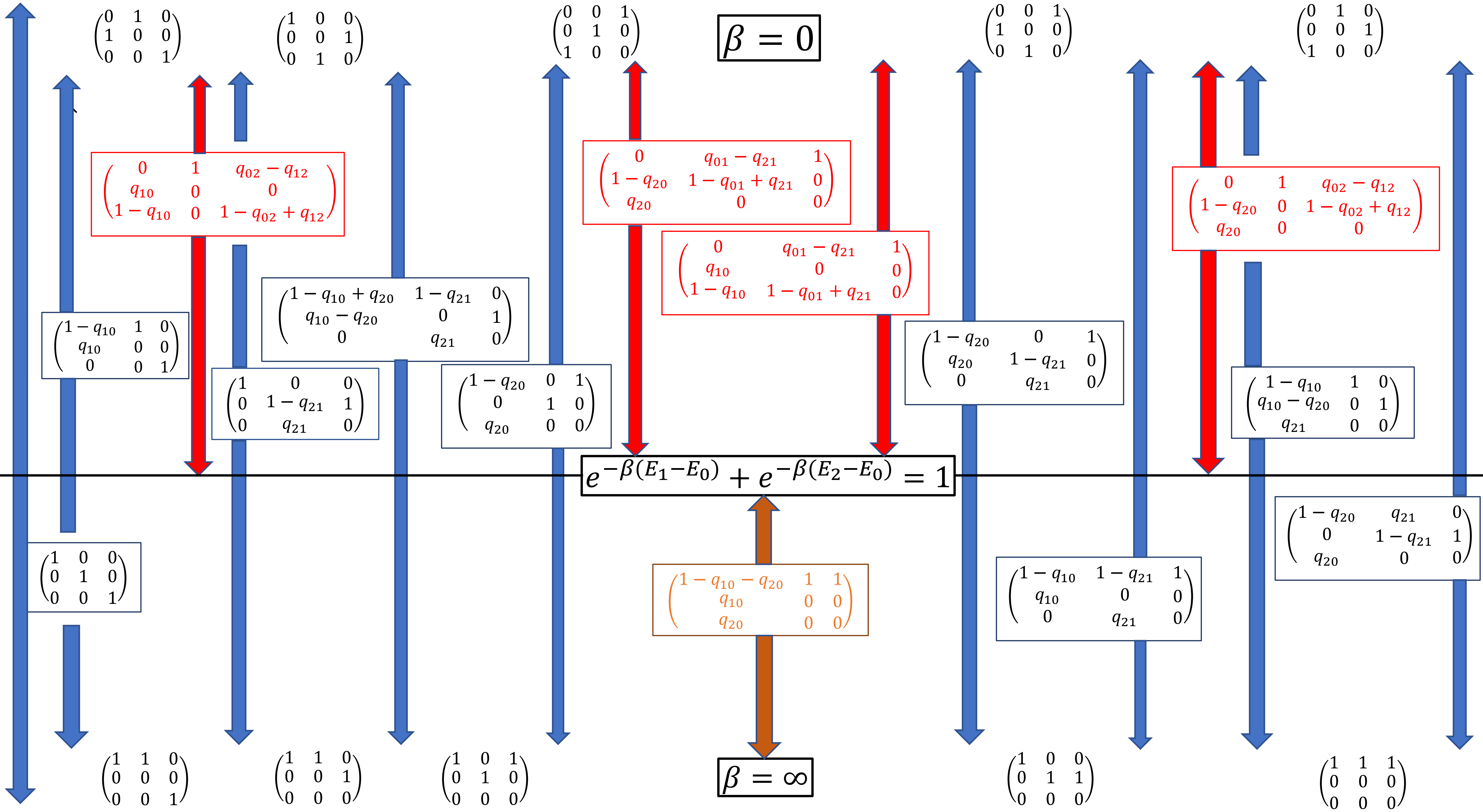
$$p = \begin{pmatrix} p_0 \\ \vdots \\ p_n \\ \vdots \\ p_m \\ \vdots \\ p_{d-1} \end{pmatrix}; \partial p = \begin{pmatrix} p_0 \\ \vdots \\ q_{0n} p_n \\ \vdots \\ q_{0m} p_m \\ \vdots \\ q_{0,d-1} p_{d-1} \end{pmatrix}, \qquad T_{ext}(p) = r \Rightarrow \partial r_m = q_{0m}(q_{mn} p_n) = q_{0n} p_n = \partial p_n.$$




d = 2

$$T_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T_1 = \begin{pmatrix} 1 - q_{10} & 1 \\ q_{10} & 0 \end{pmatrix}. \qquad T = (1-\lambda) T_0 + \lambda T_1$$

$$q_{nm} = e^{-\beta(E_m - E_n)}.$$



$\beta = 0$
$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$
$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$
$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$
$\begin{pmatrix}0&1&q_{02}-q_{12}\\q_{10}&0&0\\1-q_{10}&0&1-q_{02}+q_{12}\end{pmatrix}$
$\begin{pmatrix}0&q_{01}-q_{21}&1\\1-q_{20}&1-q_{01}+q_{21}&0\\q_{20}&0&0\end{pmatrix}$
$\begin{pmatrix}0&q_{01}-q_{21}&1\\q_{10}&0&0\\1-q_{10}&1-q_{01}+q_{21}&0\end{pmatrix}$
$\begin{pmatrix}0&1&q_{02}-q_{12}\\1-q_{20}&0&1-q_{02}+q_{12}\\q_{20}&0&0\end{pmatrix}$
$\begin{pmatrix}1-q_{10}+q_{20}&1-q_{21}&0\\q_{10}-q_{20}&0&1\\0&q_{21}&0\end{pmatrix}$
$\begin{pmatrix}1-q_{10}&1&0\\q_{10}&0&0\\0&0&1\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&1-q_{21}&1\\0&q_{21}&0\end{pmatrix}$
$\begin{pmatrix}1-q_{20}&0&1\\0&1&0\\q_{20}&0&0\end{pmatrix}$
$\begin{pmatrix}1-q_{20}&0&1\\q_{20}&1-q_{21}&0\\0&q_{21}&0\end{pmatrix}$
$\begin{pmatrix}1-q_{10}&1&0\\q_{10}-q_{20}&0&1\\q_{21}&0&0\end{pmatrix}$
$e^{-\beta(E_1-E_0)} + e^{-\beta(E_2-E_0)} = 1$
$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
$\begin{pmatrix}1-q_{20}&q_{21}&0\\0&1-q_{21}&1\\q_{20}&0&0\end{pmatrix}$
$\begin{pmatrix}1-q_{10}-q_{20}&1&1\\q_{10}&0&0\\q_{20}&0&0\end{pmatrix}$
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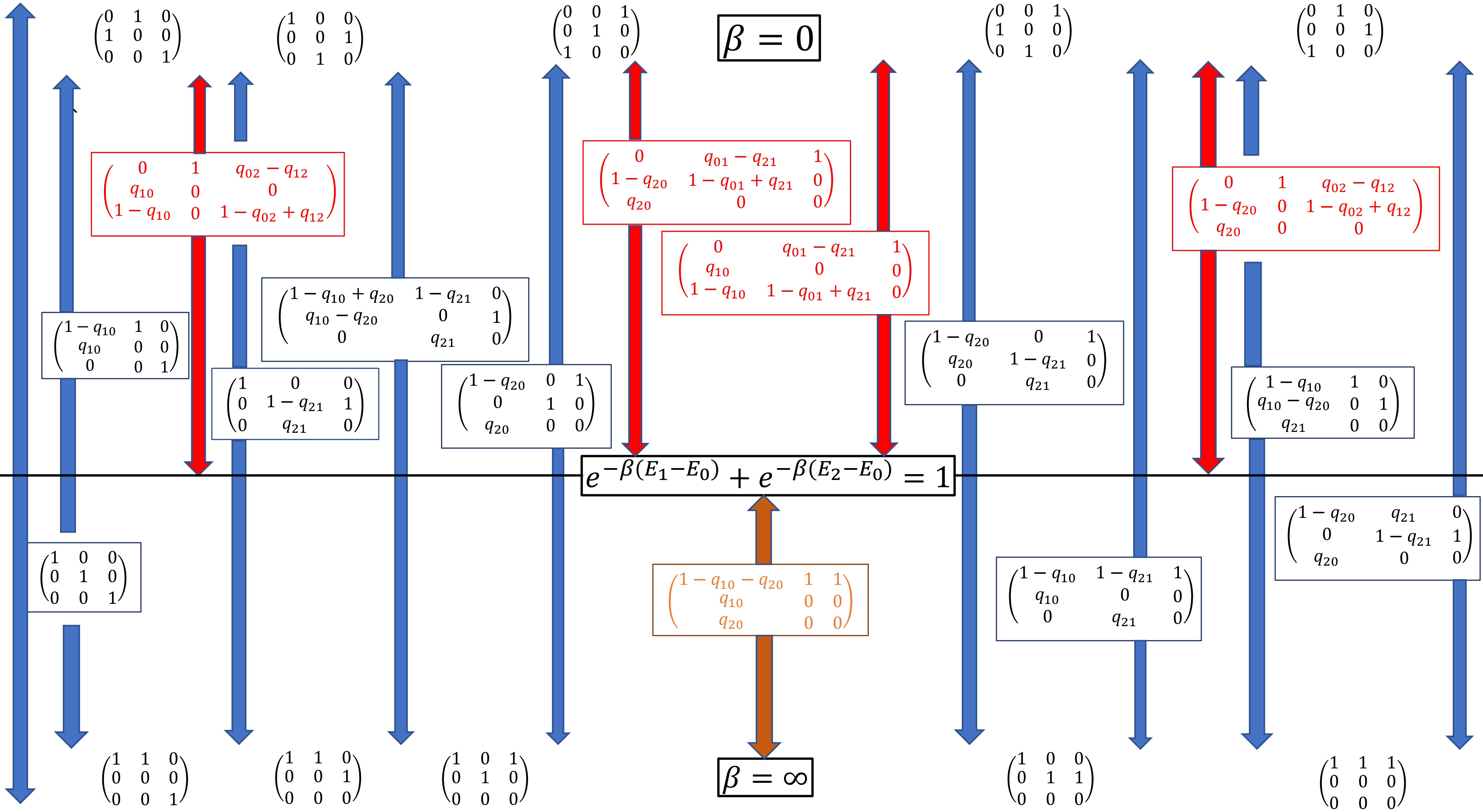
$\begin{pmatrix}1&1&0\\0&0&0\\0&0&1\end{pmatrix}$
$\begin{pmatrix}1&1&0\\0&0&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix}$
$\beta = \infty$
$\begin{pmatrix}1&0&0\\0&1&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$



$\beta = 0$
$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$
$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$
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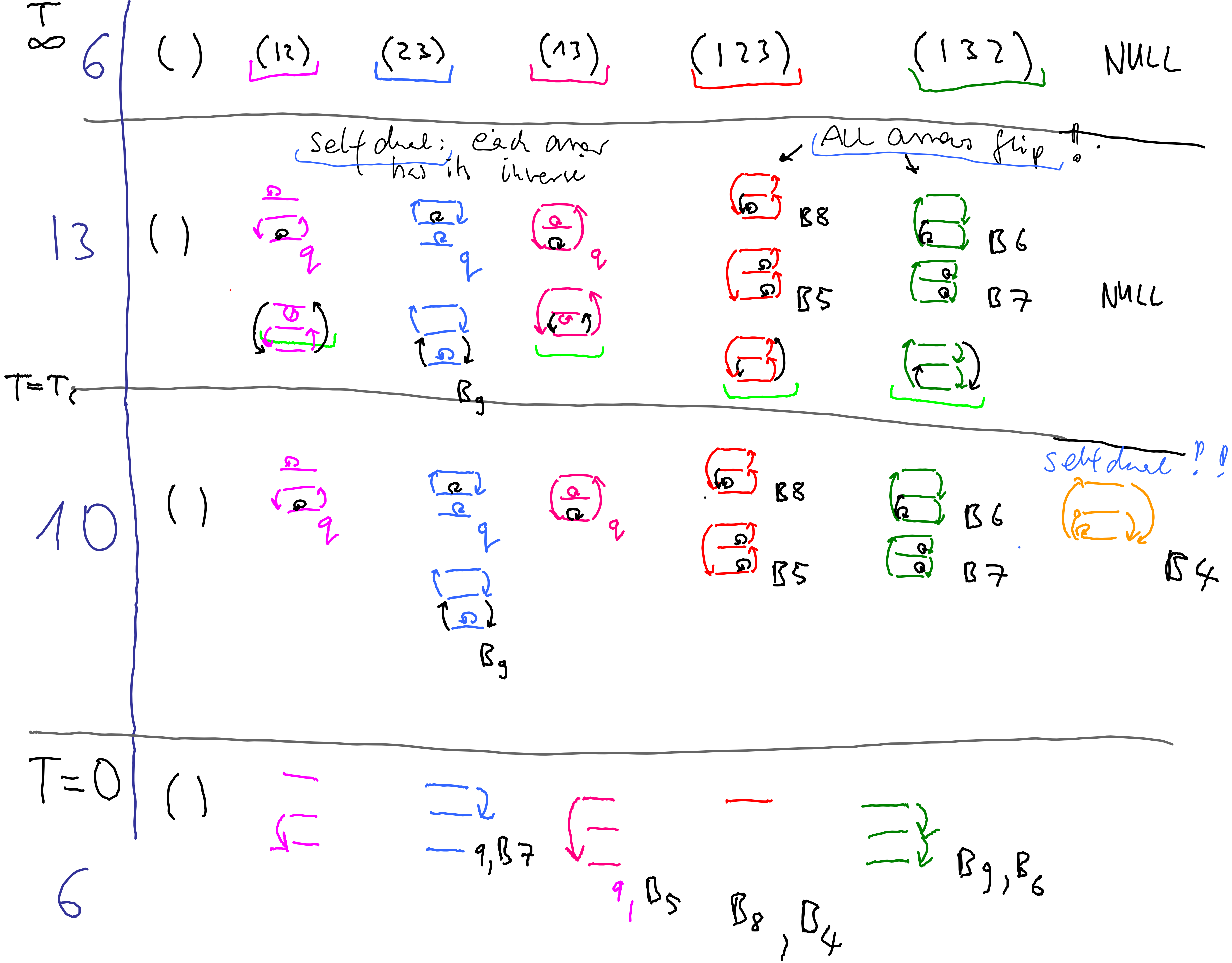
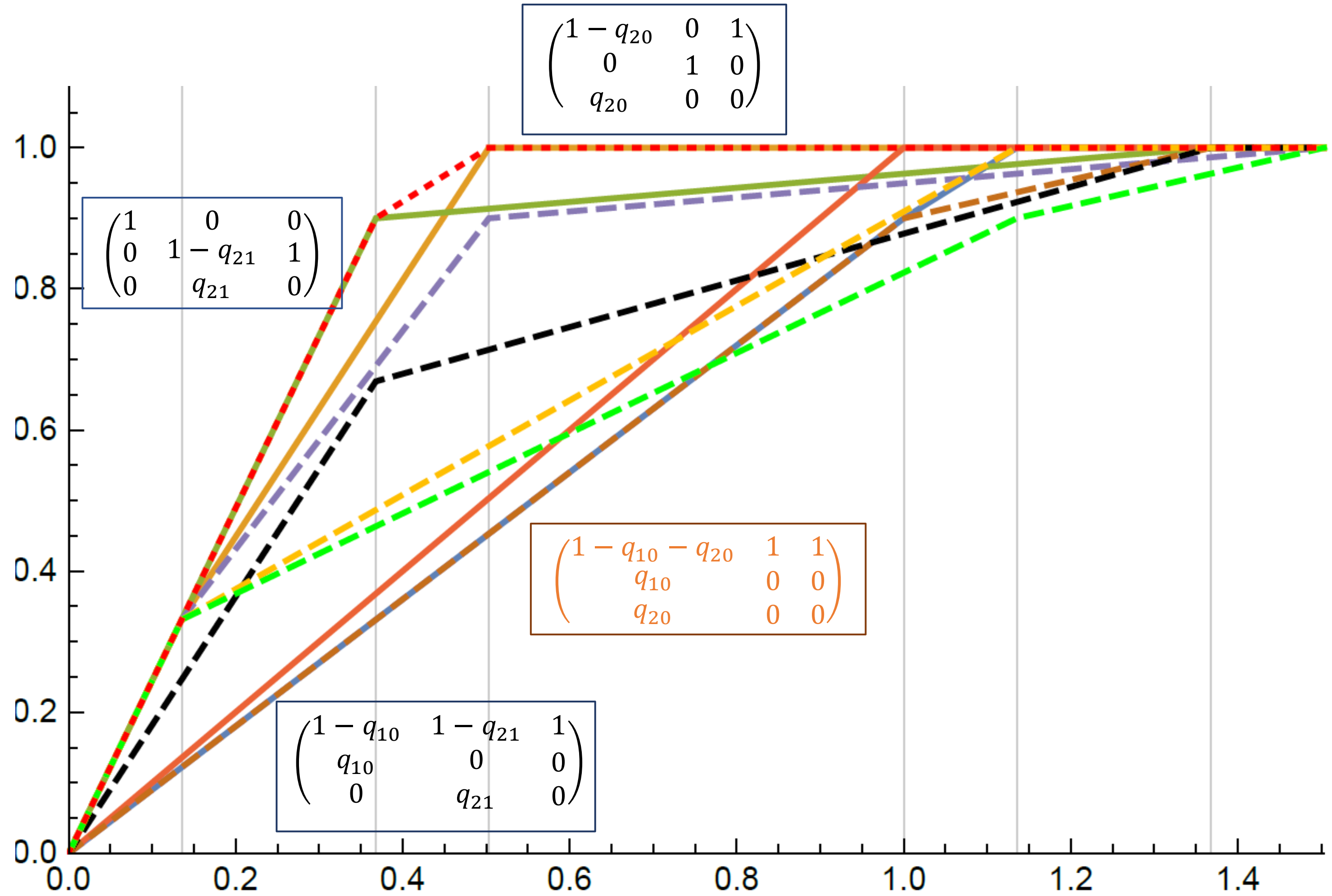


T
∞
6
( )
(12)
(23)
(13)
(123)
(132)
NULL
Self dual; each arrow has its inverse
All arrows flip!
13
( )
q
q
q
B8
B6
B5
B7
NULL
B9
T=Tc
10
( )
q
q
q
B8
B6
B5
B7
self dual!!
B4
B9
T=0
( )
q, B7
q, B5
B8, B4
B9, B6
6

# Extremal points on majorization diagrams




| Permutation of slopes for $e^{-\beta(E_1-E_0)} + e^{-\beta(E_2-E_0)} \leq 1$ | Extremal points that can be deduced |
|---|---|
| 2,1,3 ; 3,1,2 | B1, B3, B5, B6, B9 |
| 1,2,3; 3,2,1 | B1, B2, B4, B7 |

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| 2,1,3 ; 3,1,2 | B1, B3, B5, B6, B9 |
| 1,2,3; 3,2,1 | B1, B2, B7, B12,B13 |
| 2,3,1; 1,3,2 | B2, B3, B8, B10,B11 |

$q_{nm} = e^{-\beta(E_m-E_n)}$.

$q_{nm} = e^{-\beta(E_m - E_n)}$.

# Extremal points for 4 level systems

$$\begin{pmatrix} 1+q_{10}-q_{20}-q_{30} & 0 & 0 & 1-q_{13}+q_{23} \\ 0 & 0 & 1 & q_{13}-q_{23} \\ 0 & q_{21} & 0 & 0 \\ -q_{10}+q_{20}+q_{30} & 1-q_{21} & 0 & 0 \end{pmatrix}$$

Valid for $q_{20} + q_{30} > q_{10}$

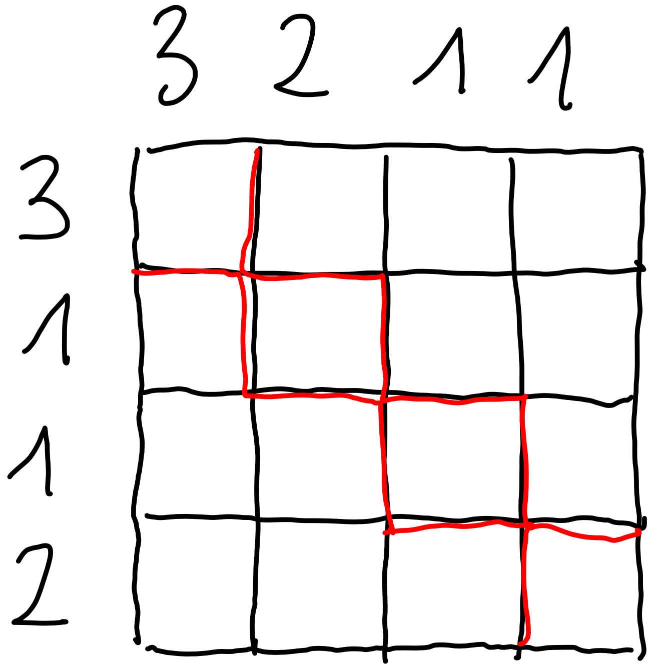
274 genuinly 4-level extremal points
Combination of elements of $q_{xn}$type in $n$-th column

How to construct an extremal point?

Go through all possible assignments of $2d-1$ non-zero elements into rows and columns (s.t. each column/row contains at least 1 such element, first column/row contains at least 2).

For each assignment, continue fixing matrix elements as long as one obtains a completely defined matrix compatible with the assignment.

E.g. for 4 levels, assignments 3,1,1,2 (rows) and 3,2,1,1 (columns):
Not to violote Gibbs: Identities above red line and single non-zero elements in a row right to the red line



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# Extremal points for 4 level systems

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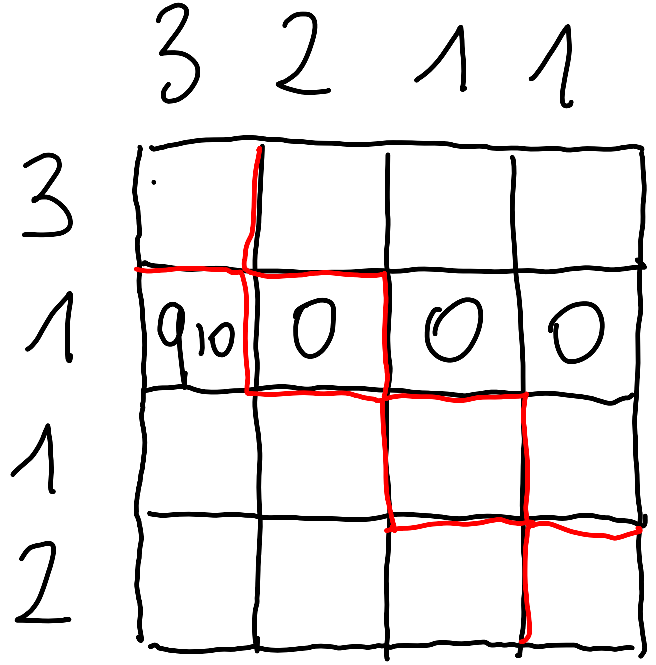

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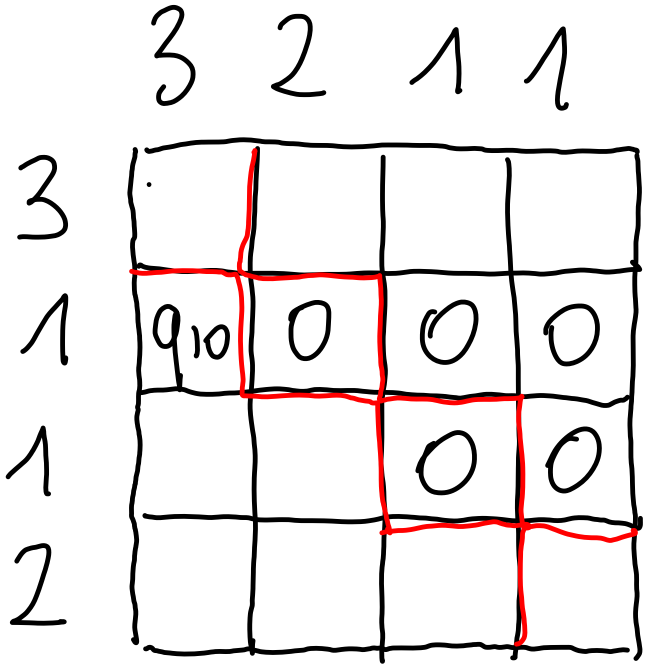

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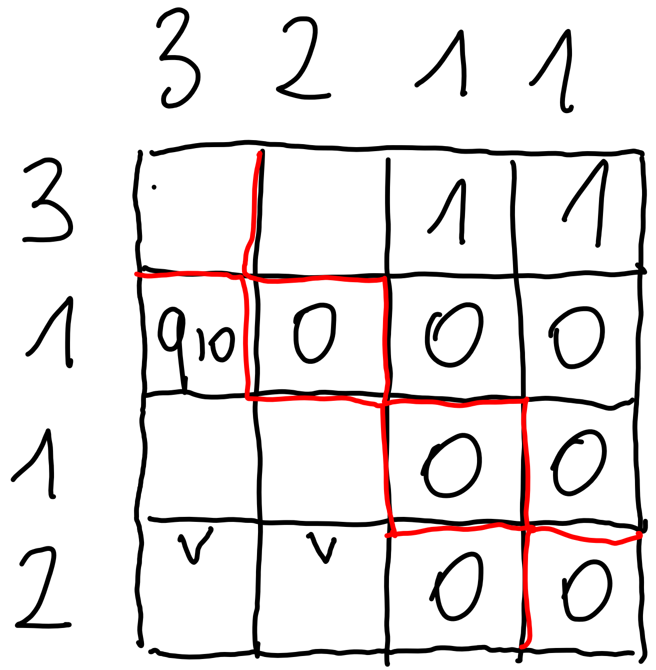
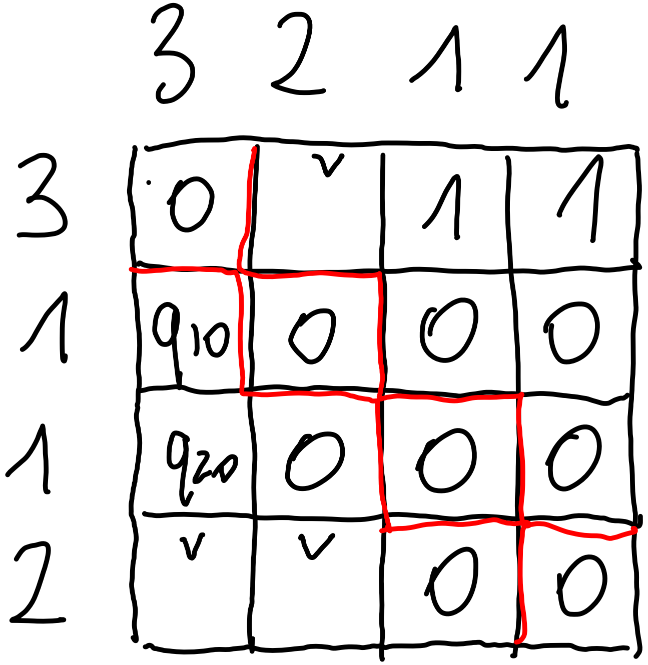
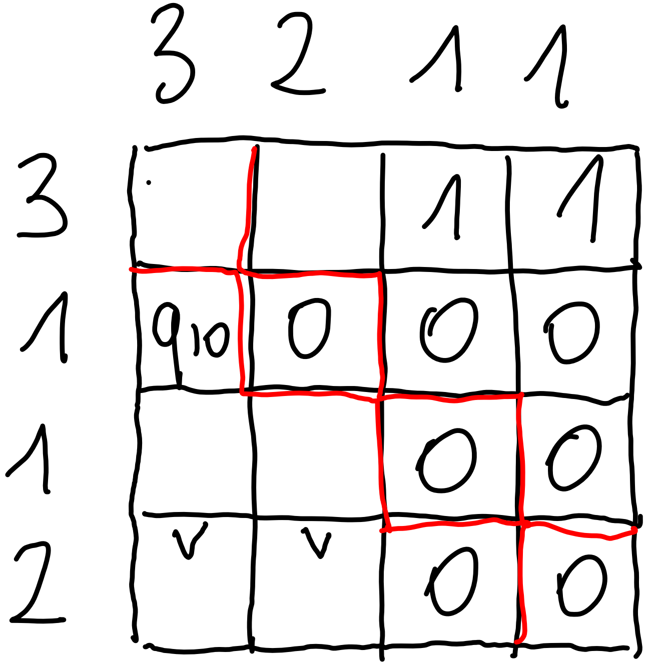


# Extremal points for 4 level systems

$$\begin{pmatrix} 1 + q_{10} - q_{20} - q_{30} & 0 & 0 & 1 - q_{13} + q_{23} \\ 0 & 0 & 1 & q_{13} - q_{23} \\ 0 & q_{21} & 0 & 0 \\ -q_{10} + q_{20} + q_{30} & 1 - q_{21} & 0 & 0 \end{pmatrix}$$

Valid for $q_{20} + q_{30} > q_{10}$



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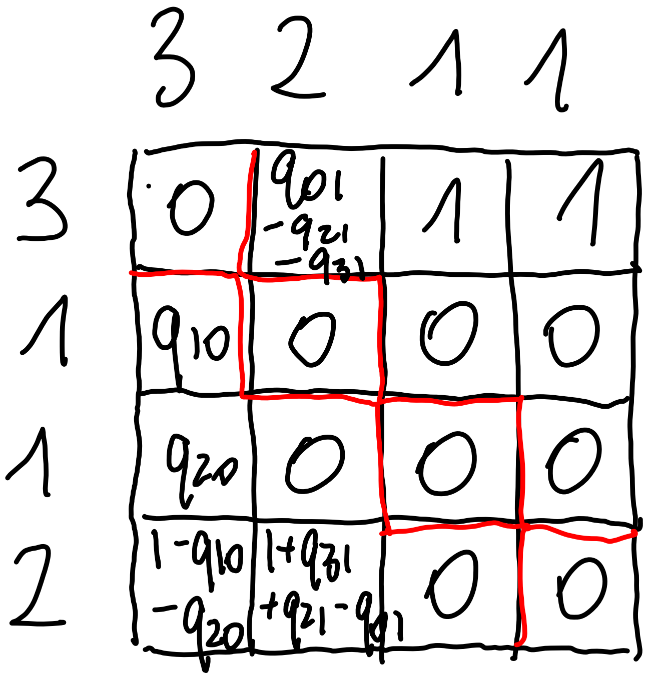


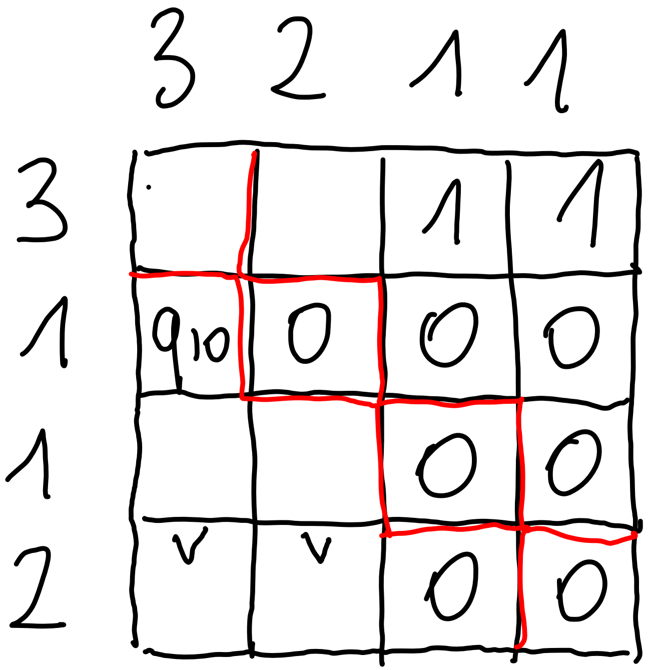

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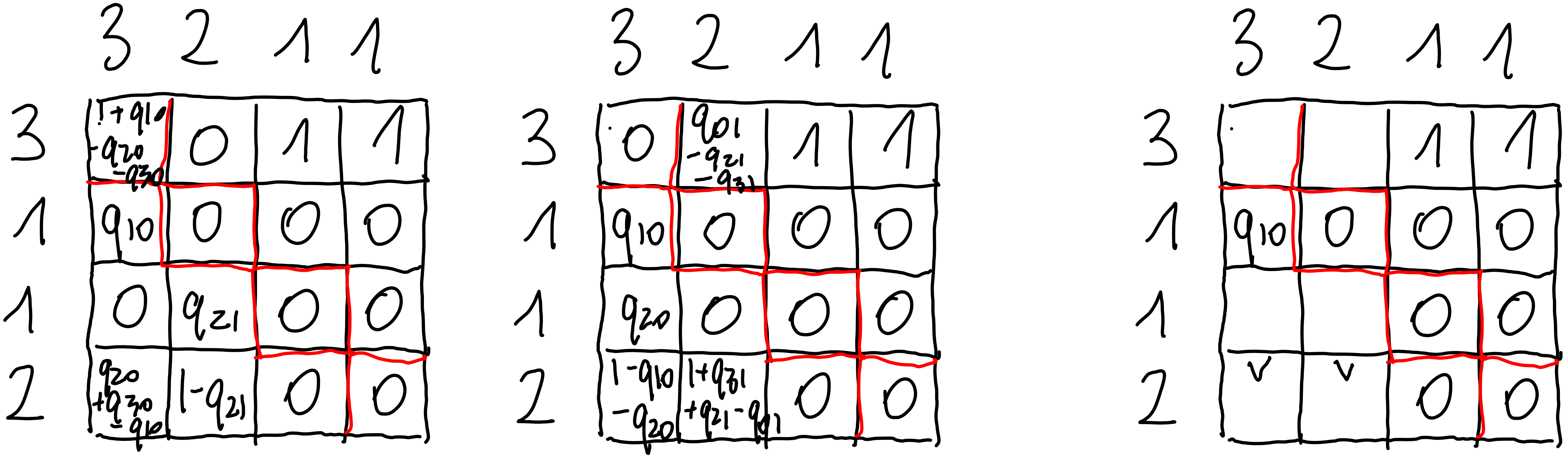

An algorithm to generate extremal points of thermal processes for d-level systems?

Structure of an extremal point?

What is the temperature dependence of the geometry of the set?

Can extermal points be always obtained from majorization diagrams?