# A classical model of continuous information extraction for stochastic thermodynamics


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Quantum Thermodynamics and Resource Theory
9-11 May 2017 Paris

# Acknowledgments

Side project with crucial input from **Irénée Frérot** and **Marti Perarnau Llobet**, using techniques learned with **Denis Bernard** and **Michel Bauer**.

Done between **ENS** (2 years ago) and **MPQ** (2 weeks ago)

**Largely in progress**

# Objectives of the inquiry

2 kinds of objectives

**Classical stochastic thermodynamics with information flows**

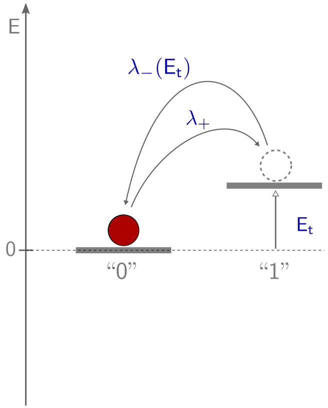
- Develop a theory of continuous information flow
- Extend optimal work extraction strategies in this context

**Quantum stochastic thermodynamics**

- Develop a transparent classical counterpart of quantum trajectories
- Get a hint for good definitions

# Model considered

A **classical** 2-level system with tunable energy




**Markov process** $M_t \in \{0, 1\}$ with energy dependent jump rates:

$$\lambda_+ = \lambda$$
$$\lambda_- = \lambda e^{-\beta E_t}$$

With $p_t = \mathrm{Prob}[M_t = 1]$ the master equation reads:

$$\partial_t p_t = -\lambda_- \, p_t + \lambda_+ \, (1 - p_t)$$

The equilibrium probability is **Boltzmann**:

$$p_t \underset{t \to +\infty}{\longrightarrow} p_{eq} = \frac{e^{-\beta E_t}}{1 + e^{-\beta E_t}}$$

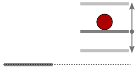
# Thermodynamical quantities

- Internal Energy:

$$U_t = M_t \times E_t$$

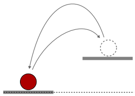
- Work:

$$\delta W_t = M_t \, dE_t$$



- Heat:

$$\delta Q_t = E_t \, dM_t$$



**First law**

$$dU_t = \delta W_t + \delta Q_t$$

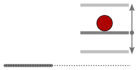
# Thermodynamical quantities – *averages*

- Internal Energy:

$$\mathbb{E}[U_t] = \mathbb{E}[M_t] \times E_t = p_t \times E_t$$

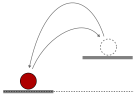
- Work:

$$\mathbb{E}[\delta W_t] = \mathbb{E}[M_t]\, dE_t = p_t\, dE_t$$



- Heat:

$$\mathbb{E}[\delta Q_t] = E_t\, \mathbb{E}[dM_t] = E_t\, dp_t$$



**First law – average**

$$\mathbb{E}[dU_t] = \mathbb{E}[\delta W_t] + \mathbb{E}[\delta Q_t]$$

# Standard questions

**Converting information into work** $\mathbb{E}[\Delta W] = -\frac{1}{\beta} \cdot \Delta I$

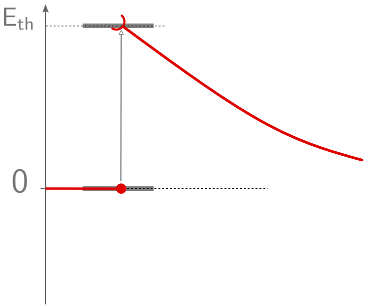
Given a certain initial probability $p_0 \neq p_{eq}$ on $M_0$, how much work can one extract:

- In infinite time?
- In finite time?
- With what protocol $\{E_t\}$?

# Infinite time

**Optimal "Landauer" protocol:**

- Apply an instant **quench** so that $p_0$ is thermal for some energy $E_{th}$
- **Slowly** bring the energy $E_t$ back to $0$.




$$\mathbb{E}[\Delta W] = \underbrace{p_0 \cdot (E_{th} - 0)}_{quench} + \underbrace{\int_0^{+\infty} p_t \, dE_t}_{adiabatic}$$

$$= -\frac{1}{\beta} p_0 \log \left[ \frac{p_0}{1 - p_0} \right] + \int_{E_{th}}^{0} p(E) \, dE$$

$$= \frac{1}{\beta} \left[ p_0 \log(p_0) + (1 - p_0) \log(1 - p_0) - \log(2) \right]$$

$$= \frac{1}{\beta} \Delta S = -\frac{1}{\beta} \Delta I$$

# Finite time[1]

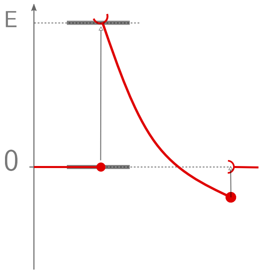
**Objective**

Find:

$$\min_{\{E\}} \mathbb{E}[\Delta W] = \min_{\{E\}} \int_0^{t_f} p_t(E)\, dE_t$$

under the constraint $E_{t_f} = 0$ and $\partial_t p_t = -\lambda e^{-\beta E_t}\, p_t + \lambda\,(1 - p_t)$

Standard optimal (non-stochastic) control problem.




Solved with **Euler-Lagrange** equation
$\Rightarrow$ transcendental 2-jump solution

[1]see *e.g.* Bauer *et al* J. Stat. Mech. (2014) P09010 and Esposito *et al* (2010) EPL **89** 20003

# Local conclusion

A given amount of **information** can be used to extract **work** which is **on average**:

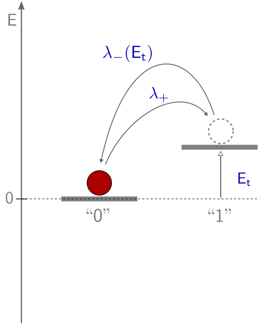
$=$ Landauer bound for infinite time

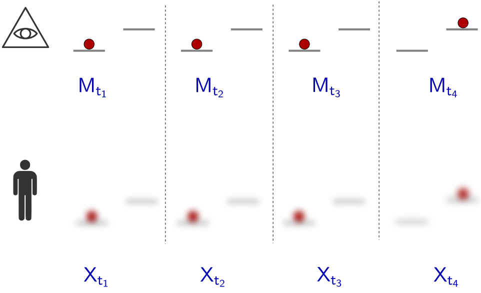
$\leq$ Landauer bound for finite time

In any case it is **single-shot**.

# A hidden Markov model[2] (HMM)

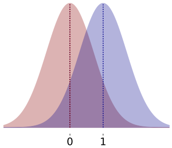
**Imperfect observations** of a Markov process



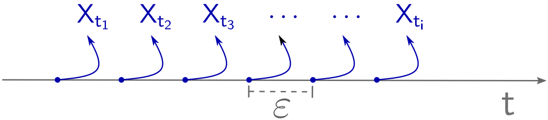

For example:

$$\mathbb{P}[X_{t_i}|M_{t_i}] \propto \exp\left(-\varepsilon \cdot \frac{(X_{t_i} - M_{t_i})^2}{2}\right)$$




[2] see *e.g.* Bechhoefer (2015) New J. Phys. **17** 075003

# A hidden Markov model

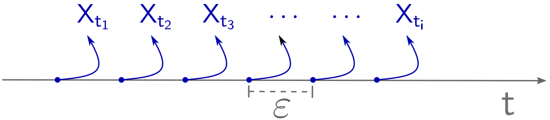



We are interested in the **filtered** probability:

$$\vec{p}_t = \mathbb{E}[M_t | X_{t_1}, \cdots, X_{t_i}, \; t_i \leq t < t_{i+1}]$$

useful for the purpose of **control**.

# A hidden Markov model

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Subtlety: we can define a **fluctuating** "work" in real time $\vec{W}_t = \int_0^t \vec{p}_u dE_u$ s.t.:

- $\mathbb{E}[\vec{W}_t] = \mathbb{E}[W_t]$
- but $\vec{W}_t$ is **NOT** the best estimate of $W_t$
- the fluctuations of $\vec{W}_t$ are meaningless

# Continuum limit

Go from **discrete weak** classical measurements to **continuous measurements**. Take

- Infinitely short time between measurements $\propto \varepsilon \to 0$
- Infinitely bad resolution $\propto \sqrt{\varepsilon} \to 0$

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One can show:

**Continuous limit**

The **filtered probability** verifies:

$$d\vec{p}_t = \underbrace{-\lambda_+(E)\vec{p}_t\, dt + \lambda_-(E)(1-\vec{p}_t)\, dt}_{\text{standard master equation}} + \underbrace{\sqrt{\gamma}\, \vec{p}_t(1-\vec{p}_t)\, dB_t}_{\textbf{information acquisition}}$$

The **"measurement"** signal:

$$X_t = \vec{p}_t + \frac{1}{\sqrt{\gamma}} \frac{dB_t}{dt}$$

where $B_t$ is a Wiener process (Brownian motion), in Itô convention.

Link with stochastic control theory: Kushner-Stratonovich filtering equation.
Link with quantum mechanics: Belavkin equation in the fully diagonal case.

# Comments

$$d\vec{p}_t = \underbrace{-\lambda_+(E)\vec{p}_t\,dt + \lambda_-(E)(1-\vec{p}_t)\,dt}_{\text{standard master equation}} + \underbrace{\sqrt{\gamma}\,\vec{p}_t(1-\vec{p}_t)\,dB_t}_{\textbf{information acquisition}}$$

- information extraction is a universal non-linear stochastic term of zero mean
- continuous but not differentiable –careful!–
- $B_t$ has nothing to do with thermal processes
- the measurement has no influence on the dynamics, *i.e.* with $p_t = \mathbb{E}[\vec{p}_t]$:

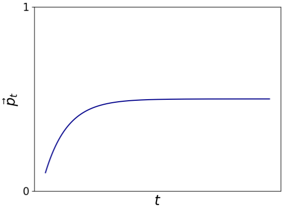
$$\partial_t p_t = -\lambda_-\, p_t + \lambda_+\,(1-p_t)$$

at least as long as $E_t$ does not depend on $\{X_u\}$.

# Basic behavior

$$d\vec{p}_t = -\lambda_+(E)\vec{p}_t\,dt + \lambda_-(E)(1-\vec{p}_t)\,dt$$

standard master equation

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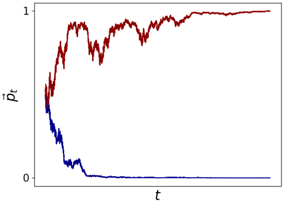
**information acquisition**



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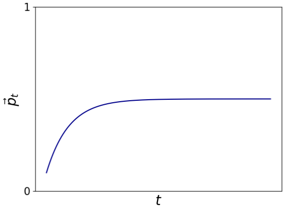


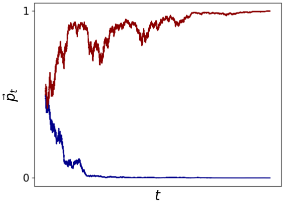


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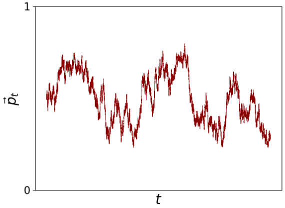
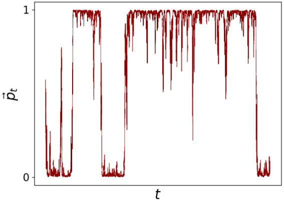



$\gamma = 1$




$\gamma = 100$

# Thermodynamical quantities

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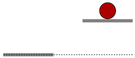
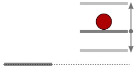


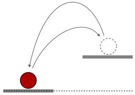
- "Work":

$$\delta \vec{W}_t = \underbrace{\vec{p}_t \, dE_t}_{\text{backward Itô}}$$



- "Heat":

$$\delta \vec{Q}_t = \underbrace{E_t \, d\vec{p}_t}_{\text{forward Itô}}$$



**First law**

$$\text{formal } d\vec{U}_t = \delta \vec{W}_t + \delta \vec{Q}_t$$

$$\begin{aligned} \text{true average } d\mathbb{E}[\vec{U}_t] &= \mathbb{E}[\delta \vec{W}_t] + \mathbb{E}[\delta \vec{Q}_t] \\ &= \mathbb{E}[\delta W_t] + \mathbb{E}[\delta Q_t] \end{aligned}$$

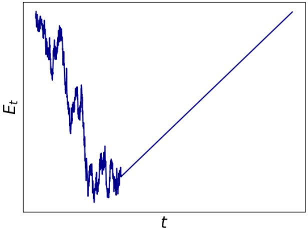
# Infinite time

**Protocol**

- Keep the energy $E_t$ such that $\vec{p}_t$ is thermal for this energy:

$$d\vec{p}_t = \underbrace{-\lambda_+(E)\vec{p}_t\,dt + \lambda_-(E)(1-\vec{p}_t)\,dt}_{=0} + \sqrt{\gamma}\,\vec{p}_t(1-\vec{p}_t)\,dB_t$$

  $\Rightarrow$ effectively cancels the jumps (in the probability),
  $\Rightarrow$ drags the (forward) probability $\vec{p}_t$ towards 0 or 1
- Wait some time $\Delta t$ or wait to reach some threshold $\tilde{E}$.
- Stop measuring and slowly bring the energy $E_t$ back to 0.



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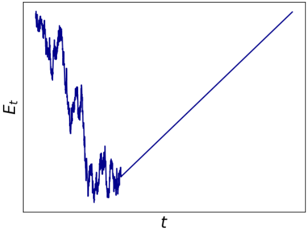


The Landauer bound is reached exactly.
Non trivial:
Work and Information exchange terms come from **Itô's lemma**; no non-trivial theory with smooth evolution.

# Real-time

**Hamilton-Bellman-Jacobi** equation...

**seems very non trivial**, any idea?

# Classical comments

- It is possible to construct theories with continuous information flow
- Information flow appears as a **universal** non-linear non-differentiable term in the master equation
- Thermodynamical quantities seem to fluctuate because of **surprise**
- These fluctuations are meaningless

# Quantum comments

Continuous quantum trajectories are very similar (identical in the diagonal case):

$$d\rho_t = \mathcal{L}(\rho_t)\, dt + \{\mathcal{O}\rho + \rho\mathcal{O} - 2\rho \mathrm{tr}\mathcal{O}\rho\}\ dB_t$$

[3] Elouard *et al.* npj Quantum Information **3**, 9 (2017) — Alonso, Lutz, Romito PRL **116**, 080403 (2016)

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**Work** and **heat** are usually[3] defined this way:

$$\underbrace{d(\mathrm{tr}H_t\rho_t)}_{dU_t} = \underbrace{\mathrm{tr}(H_t\, d\rho_t)}_{\delta Q_t} + \underbrace{\mathrm{tr}(dH_t\, \rho_t)}_{\delta W_t}$$

which correspond to $\delta\vec{W}_t$ and $\delta\vec{Q}_t$. **But**:

- "Forward" stochastic quantities are useful for computations but not necessarily physically meaningful.
- Not everything that is non-unitary is heat.

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$\Rightarrow$ use some correspondence principle to guide the quantum ?

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# Conclusion

1. Information flows can be continuous (non-differentiable) and take a universal form
2. They can be used for work extraction via **feedback control**:
   - infinite time is manageable, information flow is digested optimally
   - finite time is a hard optimization problem
3. Classical $+$ continuous information flow $\sim$ continuous quantum trajectories